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INTRODUCTION TO THE APPLICATION OF VON ZEIPEL'S METHOD

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INTRODUCTION TO THE APPLICATION OF VON ZEIPEL'S METHOD

by

William J. Wickless, Jr.

INTRODUCTION

To understand the application of Von Zeipel's method in the case of a satellite orbiting under the influence of a gravitational field plus small perturbating forces, it is first necessary to summarize the development of several concepts of classical mechanics. Therefore, the first portion of this paper will be a brief resume of material which may be found in greater detail and slightly modified form in Chapters 7-8 of Classical Dynamics of Particles and Systems by Jerry B. Marion, Dept. of Physics, University of Maryland. A good knowledge of ordinary differential and integral calculus is assumed.

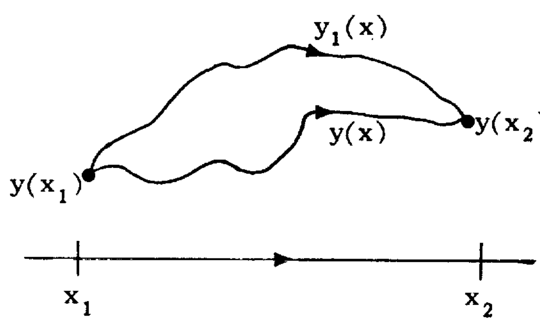
I. AN INTRODUCTION TO THE CALCULUS OF VARIATIONS

The basic problem of the calculus of variations is determining the unknown function $y(x)$ describing a path between two fixed end points we call $y(x_1)$ and $y(x_2)$ such that

$$I = \int_{x_1}^{x_2} f\,[y(x),\ y'(x),\ x]\ dx$$

takes on a maximum or minimum. f is a given function of the functions $y(x)$, $y'(x)$ and the independent variable x and the limits of integration are fixed. That is, if we wish to minimize I (say), we wish to find a function $y(x)$ such that if $y_1(x)$ is any other continuous function such that $y_1(x_1) = y(x_1)$, $y_1(x_2) = y(x_2)$ – any other path between $y(x_1)$ and $y(x_2)$ – then:

$$\int_{x_1}^{x_2} f\left[y(x),\ y'(x),\ x\right] dx \leq \int_{x_1}^{x_2} f\left[y_1(x),\ y_1'(x),\ x\right] dx$$



(All functions will be assumed to be differentiable to any needed order)

Two possible paths between $y(x_1)$ and $y(x_2)$ are sketched above.

We will consider families of functions, giving possible paths between two fixed points $y(x_1)$ and $y(x_2)$, indexed by a parameter α, α running over a suitable segment of the real line. That is, we will consider families of the form

$$\left\{y(\alpha,\ x)\right\}_{\alpha \in [a,\ b]}^{x \in [x_1,\ x_2]} \quad \text{with}$$

$$\left.\begin{cases} y(\alpha, x_1) = y(x_1) = \text{constant} \\ \\ y(\alpha, x_2) = y(x_2) = \text{constant} \end{cases}\right\} \begin{array}{l}\text{for all} \\ \alpha \epsilon\ [a, b].\end{array}$$

Then the integral I becomes a function of the parameter α:

$$I(\alpha) = \int_{x_1}^{x_2} f\,[y(\alpha, x),\ y'(\alpha, x),\ x]\ dx.$$

For example

$$I(\alpha) = \int_0^{2\pi} x\ (\alpha \sin x)\ dx$$

$$\alpha \epsilon\ [1, 2]$$

where

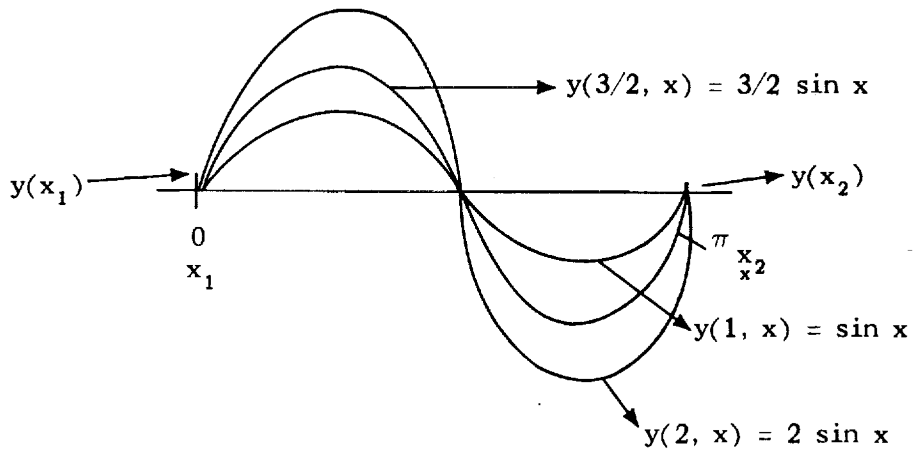
$$y(\alpha, x) = \alpha \sin x$$

$$f\,[x,\ y\,(\alpha,\ x),\ y'\ (\alpha,\ x)] = xy\,(\alpha,\ x)$$

$$x_1 = 0,\ x_2 = 2\pi$$

$$y(x_1) = 0 \quad y(x_2) = 0$$

NOTE: $\alpha \sin(0) = 0$; $\alpha \sin 2\pi = 0$ for all $\alpha \epsilon\ [1, 2]$; $\{\alpha \sin x\}$ represents the family of paths between the pts $(0, 0) \equiv y(x_1)$, $(2\pi, 0) \equiv y(x_2)$ shown at top of following page.



Now if the integral so written is to have an extremum along some path $y(\alpha_0, x)$

$$\alpha_0 \epsilon [a, b]$$

then

$$\left.\frac{\partial I(\alpha)}{\partial \alpha}\right|_{\alpha=\alpha_0} = 0.$$

This means

$$\left.\frac{\partial}{\partial \alpha} \int_{x_1}^{x_2} f\,[y(\alpha, x),\; y'(\alpha, y),\; x]\; dx\right|_{\alpha=\alpha_0} = 0.$$

since x is independent of α, the partial may be taken under the integral sign:

$$\int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha}\right] dx \Bigg|_{\alpha=\alpha_0} = 0.$$

Now the above equation may be integrated by parts (see (1), Sec. 7-3) to obtain

$$\int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right]\frac{\partial y}{\partial \alpha}\, dx \Bigg|_{\alpha=\alpha_0} = 0.$$

As f is assumed to be an arbitrary function of $y(\alpha, x)\ y'(\alpha, x)\ x$, the integrand itself must vanish:

$$\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right]\frac{\partial y}{\partial \alpha}\Bigg|_{\alpha=\alpha_0} = 0$$

for all $x \in [x_1, x_2]$.

Also since $\{y(\alpha, x)\}$ is an arbitrary family of paths, in general

$$\frac{\partial y(\alpha, x)}{\partial \alpha}\Bigg|_{\alpha=\alpha_0} \neq 0 \quad \text{along the curve}$$

$$y = y(\alpha_0, x)$$

$$x \in [x_1, x_2]$$

Now if $x(t)$ is the path followed by the particle, the Euler-Lagrange equation must be satisfied, i.e.

$$\frac{\partial L}{\partial x(t)} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}(t)} = 0 .$$

Now

$$\frac{\partial L}{\partial x(t)} = - K x(t); \quad \frac{\partial L}{\partial \dot{x}(t)} = m \dot{x}(t)$$

$$\therefore - K x(t) = \frac{d}{dt} [m \dot{x}(t)]$$

or

$$\ddot{x}(t) + w_0^2 x(t) = 0 \quad \text{with} \quad w_0 \equiv \sqrt{K/m} .$$

This differential equation may be solved (see any elementary text) to obtain

$$x(t) = A \sin (w_0 t + \delta)$$

where $A + \delta$ are constants of integration to be fitted to the initial data – in this case $A = h$, $\delta = \pi/2$.

Now an important thing to notice is that the Lagrangian is a scalar function (being the difference of kinetic and potential energies – both scalar functions). Thus we can, if desired, transform one co-ordinate variable $x(t)$ to a new variable $q(t)$ under some co-ordinate transformation ϕ and we have

$$L(x(t), \dot{x}(t), t) = \bar{L}(q(t), \dot{q}(t), t).$$

(This is actually the defining property of a scalar function).

The Euler-Lagrange equation is still valid:

$$\frac{\partial \bar{L}}{\partial q(t)} - \frac{d}{dt}\frac{\partial \bar{L}}{\partial \dot{q}(t)} = 0.$$

Now as one final generalization, it is clear the original analysis could have been carried out to treat the case of a particle with more than one degree of freedom. The Lagrangian then would have been a function of several co-ordinate variables, their derivatives, and possibly the time, i.e.

$$L = L(x_1(t) \cdots x_n(t), \dot{x}_1(t) \cdots \dot{x}_n(t), t).$$

We then could proceed in a similar fashion [see (1), Sec. 7-7] to obtain, instead of the single equation

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$$

the system of equations:

$$\frac{\partial L}{\partial xj} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}j} = 0$$

$$j = 1, 2 \cdots N.$$

Now as this Lagrangian is still a scalar function, we may make if desired, a co-ordinate transformation ϕ from the position variables $\{x_1(t) \cdots x_n(t)\}$ to some new set of variables $\{q_1(t) \ldots . q_n(t)\}$ and we still have:

$$\bar{L}(q_1 \cdots\cdots q_n, \dot{q}_1 \cdots\cdots \dot{q}_n, t) = L(x_1 \cdots x_n, \dot{x}_1 \cdots \dot{x}_n, t).$$

And also

$$\frac{\partial \bar{L}}{\partial qj} - \frac{d}{dt}\frac{\partial \bar{L}}{\partial \dot{q}j} = 0. \qquad j = 1 \ldots n \quad (a)$$

It is important to note that the new variables may be chosen as a set of <u>any N functions</u> $q_1(t) \cdots\cdots q_n(t)$ at all as long as the Lagrangian may be expressed as $L = L(q_1(t) \cdots q_n(t), \dot{q}_1(t) \cdots \dot{q}_n(t)\ t)$. In a real problem, a set of variables $q_1(t) \cdots q_n(t)$ are usually chosen so that L may be easily expressed as $\bar{L}(q_1 \cdots q_n\ \dot{q}_1 \cdots \dot{q}_n\ t)$. Then the system (a) is integrated, giving $qj = qj(t$, constants of integration) $j = 1 \cdots n$ (b).

Finally the transformation ϕ is inverted and ϕ^{-1} is applied to (b), yielding

$$xj = xj(t) \quad j = 1 \cdots\cdots N.$$

The vector $\vec{r}(t) = (x_1(t) \cdots\cdots x_n(t))$ is then the position vector of the particle under consideration.

$$\int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha}\right] dx \Bigg|_{\alpha=\alpha_0} = 0 .$$

Now the above equation may be integrated by parts (see (1), Sec. 7-3) to obtain

$$\int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right] \frac{\partial y}{\partial \alpha} dx \Bigg|_{\alpha=\alpha_0} = 0 .$$

As f is assumed to be an arbitrary function of $y(\alpha, x)\ y'(\alpha, x)\ x$, the integrand itself must vanish:

$$\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right] \frac{\partial y}{\partial \alpha}\Bigg|_{\alpha=\alpha_0} = 0$$

for all $x \in [x_1, x_2]$.

Also since $\{y(\alpha, x)\}$ is an arbitrary family of paths, in general

$$\frac{\partial y(\alpha, x)}{\partial \alpha}\Bigg|_{\alpha=\alpha_0} \neq 0 \quad \text{along the curve}$$

$$y = y\ (\alpha_0, x)$$

$$x \in [x_1, x_2]$$

therefore

$$\frac{\partial f}{\partial y(\alpha_0, x)} - \frac{d}{dx} \frac{\partial f}{\partial y'(\alpha_0, x)} = 0$$

for all $x \epsilon [x_1, x_2]$.

This is known as the Euler-Lagrange equation.

II. APPLICATION OF THE EULER-LAGRANGE EQUATION TO PHYSICAL SYSTEMS

It is known that if a particle is allowed to move between two fixed pts in the plane $x(t_1)$ and $x(t_2)$ in the time interval $[t_1, t_2]$ in any of the possible paths given by the family of curves $\{x(\alpha, t)\}$ $t \epsilon [t_1, t_2]$

$\alpha \epsilon [a, b]$

it will actually move in the path $x = x(\alpha_0, t)$ for which

$$\int_{t_1}^{t_2} (T - U)\, dt$$ is a relative extreme. (Note that this is a local condition.)

(see (1), Sec. 8.3)

Here T is the kinetic energy of the particle at the point $x(\alpha, t)$ - $T = 1/2\, m\, \dot{x}^2(\alpha, t)$ – and U is the potential energy at the same pt $U = U(x(\alpha, t))$.

$\therefore$ along the path of the particle's motion $\bar{x}(t) = x(\alpha_0, t)$, the Euler-Lagrange equation must be satisfied by the quantity $L = (T - U)$. L is called the Lagrangian of the system.

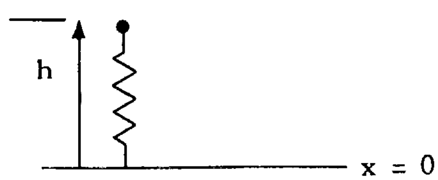
Thus we have

$$\frac{\partial L}{\partial \bar{x}(t)} - \frac{d}{dt} \frac{\partial L}{\partial \bar{x}'(t)} = 0 .$$

If the Lagrangian of a system is known, the above equation may be integrated giving the path of the particles motion $x = \bar{x}(t)$.

EXAMPLE: Determination of the Equation of Motion for a Harmonic Oscillator

Let a particle of mass m be attached to a spring with spring constant K and relaxed length 0. Choose a co-ordinate system such that the point of equilibrium of the spring is at $x = 0$. Let the mass be pulled to a distance h above the $x = 0$ level and released at a time $t = 0$. Find $x(t)$ (the bar is hereafter dropped for convenience)



Now for any time t, we have $T = 1/2\, m\, \dot{x}^2(t)$ and $U = 1/2\, K\, x^2(t)$.

so $L = T - U = 1/2\, m\, \dot{x}^2(t) - 1/2\, K\, x^2(t)$.

Now if $x(t)$ is the path followed by the particle, the Euler-Lagrange equation must be satisfied, i.e.

$$\frac{\partial L}{\partial x(t)} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}(t)} = 0 .$$

Now

$$\frac{\partial L}{\partial x(t)} = - K x(t); \quad \frac{\partial L}{\partial \dot{x}(t)} = m\,\dot{x}(t)$$

$$\therefore - K x(t) = \frac{d}{dt}\,[m\,\dot{x}(t)]$$

or

$$\ddot{x}(t) + w_0^2\, x(t) = 0 \quad \text{with} \quad w_0 \equiv \sqrt{K/m} .$$

This differential equation may be solved (see any elementary text) to obtain

$$x(t) = A \sin (w_0 t + \delta)$$

where $A + \delta$ are constants of integration to be fitted to the initial data – in this case $A = h$, $\delta = \pi/2$.

Now an important thing to notice is that the Lagrangian is a scalar function (being the difference of kinetic and potential energies – both scalar functions). Thus we can, if desired, transform one co-ordinate variable $x(t)$ to a new variable $q(t)$ under some co-ordinate transformation ϕ and we have

$$L(x(t),\ \dot{x}(t),\ t) = \bar{L}(q(t),\ \dot{q}(t),\ t).$$

(This is actually the defining property of a scalar function).

The Euler-Lagrange equation is still valid:

$$\frac{\partial \bar{L}}{\partial q(t)} - \frac{d}{dt}\frac{\partial \bar{L}}{\partial \dot{q}(t)} = 0.$$

Now as one final generalization, it is clear the original analysis could have been carried out to treat the case of a particle with more than one degree of freedom. The Lagrangian then would have been a function of several co-ordinate variables, their derivatives, and possibly the time, i.e.

$$L = L(x_1(t) \cdots x_n(t),\ \dot{x}_1(t) \cdots \dot{x}_n(t),\ t).$$

We then could proceed in a similar fashion [see (1), Sec. 7-7] to obtain, instead of the single equation

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$$

the system of equations:

$$\frac{\partial L}{\partial x j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x} j} = 0$$

$$j = 1,\ 2 \cdots N.$$

Now as this Lagrangian is still a scalar function, we may make if desired, a co-ordinate transformation ϕ from the position variables $\{x_1(t) \cdots x_n(t)\}$ to some new set of variables $\{q_1(t) \ldots \ldots q_n(t)\}$ and we still have:

$$\overline{L}(q_1 \cdots\cdots q_n, \dot{q}_1 \cdots\cdots \dot{q}_n, t) = L(x_1 \cdots x_n, \dot{x}_1 \cdots \dot{x}_n, t).$$

And also

$$\frac{\partial \overline{L}}{\partial qj} - \frac{d}{dt}\frac{\partial \overline{L}}{\partial \dot{q}j} = 0. \qquad j = 1 \ldots n \quad (a)$$

It is important to note that the new variables may be chosen as a set of <u>any N functions</u> $q_1(t) \cdots\cdots q_n(t)$ at all as long as the Lagrangian may be expressed as $L = L(q_1(t) \cdots q_n(t), \dot{q}_1(t) \cdots \dot{q}_n(t)\ t)$. In a real problem, a set of variables $q_1(t) \cdots q_n(t)$ are usually chosen so that L may be easily expressed as $\overline{L}(q_1 \cdots q_n\ \dot{q}_1 \cdots \dot{q}_n\ t)$. Then the system (a) is integrated, giving $qj = qj(t$, constants of integration) $j = 1 \cdots n$ (b).

Finally the transformation ϕ is inverted and ϕ^{-1} is applied to (b), yielding

$$xj = xj(t) \quad j = 1 \cdots\cdots N.$$

The vector $\vec{r}(t) = (x_1(t) \cdots\cdots x_n(t))$ is then the position vector of the particle under consideration.

III. CONSTRUCTION OF THE HAMILTONIAN; THE CANONICAL EQUATIONS OF MOTION

Now let us consider a system in which (1) the Lagrangian is not an explicit function of the time, i.e. $L = L(q_1(t) \cdots q_n(t), \dot{q}_1(t) \cdots \dot{q}_n(t))$ and (2) the q_j's have been obtained from the x_j's by some transformation ϕ that does not explicitly involve the time.

Let us define N new quantities:

$$P_j \equiv \frac{\partial L}{\partial \dot{q}_j} \quad j = 1, 2 \cdots N. \qquad \text{(I)}$$

P_j is called the conjugate momentum associated with q_j.

Using this definition, the Euler-Lagrange equations for our system may be rewritten in the form

$$\dot{P}_j = \frac{\partial L}{\partial q_j} \quad j = 1, 2 \cdots N. \qquad \text{(II)}$$

Now let us define a new quantity

$$H \equiv \sum_j P_j \dot{q}_j - L.$$

H is called the Hamiltonian of the system, and under assumptions (1) and (2) above, it can be shown that H is equal to the total energy of the particle under consideration. $H = T + U$. (See (1), Sec. 8-8).

Now if H is considered to be a function of $q_j, P_j \quad j = 1 \cdots N$, then

$$dH = \sum_j \frac{\partial H}{\partial q_j} dq_j + \frac{\partial H}{\partial P_j} dP_j \tag{A}$$

also since

$$H = \sum_j P_j \dot{q}_j - L$$

$$dH = \sum_j P_j d\dot{q}_j + \dot{q}_j dP_j - \frac{\partial L}{\partial q_j} dq_j - \frac{\partial L}{\partial \dot{q}_j} dq_j$$

using (I) and (II), the above becomes

$$dH = \sum_j [P_j d\dot{q}_j - P_j d\dot{q}_j] + \dot{q}_j dP_j - \dot{P}_j dq_j$$

$$= \sum_j \dot{q}_j dP_j - \dot{P}_j dq_j . \tag{B}$$

Now identifying the coefficients of dP_j & dq_j in (A) and (B) we have

$$-\dot{P}_j = \frac{\partial H}{\partial q_j} \quad \dot{q}_j = \frac{\partial H}{\partial P_j} \quad j = 1, 2 \cdots N.$$

These are called the <u>cyclic equations of motion</u>. As in the case of the Euler-Lagrange equations, they may be integrated to obtain the equations of motion of the particle under consideration.

We will now consider our problem—determination of the equations of motion for a satellite orbiting under the influence of a gravitational field plus small perturbations (e.g. the effects of the earth's oblateness.) The following (sec. IV) is a summary of a portion of the material found in the paper "Notes on Von Zeipel's Method" by Giorgio E. O. Giacaglia, published at the Goddard Space Flight Center.

IV. VON ZEIPEL'S METHOD

In the case we are considering, the Hamiltonian may be expressed as a function of six variables (L, G, H, ℓ, g, h) where L, G, H are the conjugate momenta associated with ℓ, g, h respectively. These six variables are called the Delaunay variables.

The equations to be integrated for our problem then are:

$$\dot{\ell} = \frac{\partial H}{\partial L} \quad \dot{g} = \frac{\partial H}{\partial G} \quad \dot{h} = \frac{\partial H}{\partial H}$$

$$-\dot{L} = \frac{\partial H}{\partial \ell} \quad -\dot{G} = \frac{\partial H}{\partial g} \quad -\dot{H} = \frac{\partial H}{\partial h}$$

here we have just taken the cyclic equations of motion and set $L = P_1$, $G = P_2$, $H = P_3$, $\ell = q_1$, $g = q_2$, $h = q_3$.

In many cases the negative of the Hamiltonian $F = -H$ is introduced; the cyclic equations of motion for our system then become:

$$\dot{\ell} = -\frac{\partial F}{\partial L} \quad \dot{g} = -\frac{\partial F}{\partial G} \quad \dot{h} = -\frac{\partial F}{\partial H}$$

$$\dot{L} = \frac{\partial F}{\partial \ell} \quad \dot{G} = \frac{\partial F}{\partial g} \quad \dot{H} = \frac{\partial F}{\partial h}.$$

Now in solving the above system, it would greatly simplify our problem if we could transform our variables (L, G, H, ℓ, g, h) to a new set of variables $(L', G', H', \ell', g', h')$ in which the new function F could be expressed in a form not explicitly containing one of the new variables (ℓ' say), i.e.

$$F' = F'(L', G', H' - g', h')$$

then

$$\dot{L}' = \frac{\partial F'}{\partial \ell'} = 0; \quad L' = \text{constant}.$$

Here we must remember that, for the cyclic equations of motion to be valid, the new variables must satisfy

$$L' = \frac{\partial L}{\partial \dot{\ell}'} \quad G' = \frac{\partial L}{\partial \dot{g}'} \quad H' = \frac{\partial L}{\partial \dot{h}'}$$

i.e. L', G', H' must be the conjugate momenta with respect to ℓ', g', h'. (This assumption was used to derive the cyclic equations of motion).

If the above equations are satisfied, then $(L', G', H', \ell', g', h')$ are said to be a <u>canonical set</u> of variables, and the transformation connecting

the primed and unprimed variables is called a *canonical transformation* [assuming $F(L, G, H, \ell, g, h) = F'(L', G', H', - g', h')$ for all times t].

Now finding the desired canonical transformation connecting (L, G, H, ℓ, g, h) to a new set of variables ($L', G', H', \ell', g', h'$) for which $\partial F'/\partial\ell' = 0$ is equivalent to finding a function $S = S(\ell, g, h, L', G', H')$, called a *generating function*, satisfying:

$$L = \frac{\partial S}{\partial \ell} \quad G = \frac{\partial S}{\partial g} \quad H = \frac{\partial S}{\partial h}$$

$$\ell' = \frac{\partial S}{\partial L'} \quad g' = \frac{\partial S}{\partial G'} \quad h' = \frac{\partial S}{\partial H'} .$$

This function S will determine an implicit transformation connecting (L, G, H, ℓ, g, h) and ($L', G', H', \ell', g', h'$) and it is known that the new set of variables will be canonical. (See *Planetary Theory* by Brown and Shook, sec. 5-3.)

We now shall give explicit formulas for determining S and F'. Here we first employ the assumption that our satellite is orbiting in a gravitational field, subject to small perturbing forces. We assume the function $F(L, G, H, \ell, g, h)$ can be expanded in a series

$$F = F_0 + \sum_{j=1}^{\infty} F_j \lambda^j$$

where F_0 is the negative of the Hamiltonian calculated in the case of a body orbiting under the influence of gravity alone and λ is a small parameter. (For the calculation of $F_0 = \mu^2/2L^2$ where $\mu = K^2M$, with K the Gaussian constant and M the mass of the orbited body; see any intermediate textbook in celestial mechanics.)

We assume similarly:

$$F' = F_0' + \sum_{j=1}^{\infty} F_j \lambda^j$$

$$S = S_0 + \sum_{j=1}^{\infty} S_j \lambda^j$$

where S_0 is the generating function determining the identity transformation, $S_0 = L'\ell + G'g + H'h$. (In the Keplerian case the original set of variables are the ones desired; no transformation is necessary.)

Now to any order in λ, $F(L, G, H, \ell, g, h) = F'(L', G', H' - g', h')$. (Remember F' is not to be an explicit function of ℓ'.)

Look at this equation up to second order in λ, i.e.

$$F_0 + \lambda F_1 + \lambda^2 F_2 = F_0' + \lambda F_1' + \lambda^2 F_2' \qquad \text{(A)}$$

now

$$L = \frac{\partial S}{\partial \ell} = \frac{\partial S_0}{\partial \ell} + \sum_{j=1}^{\infty} \lambda^j \frac{\partial S_j}{\partial \ell} =$$

$$L' + \sum_{j=1}^{\infty} \lambda^j \frac{\partial S_j}{\partial \ell}.$$

$$g' = \frac{\partial S}{\partial G'} = \frac{\partial S_0}{\partial G'} + \sum_{j=1}^{\infty} \lambda^j \frac{\partial S_j}{\partial G'} =$$

$$g + \sum_{j=1}^{\infty} \lambda^j \frac{\partial S_j}{\partial G'}$$

with similar expressions for G, H, h'.

Substituting the above into (A), we have

$$F_0\left(L' + \sum_{j=1}^{\infty} \lambda^j \frac{\partial S_j}{\partial \ell}\right) + \lambda F_1\left(L' + \sum_{j=1}^{\infty} \lambda^j \frac{\partial S_j}{\partial \ell},\ G' + \sum_{j=1}^{\infty} \lambda^j \frac{\partial S_j}{\partial g},\right.$$

$$\left. H' + \sum_{j=1}^{\infty} \lambda^j \frac{\partial S_j}{\partial h},\ \ell,\ g,\ h\right) + \lambda^2 F_2 \left(L' + \sum,\right.$$

$$G' + \sum, \; H' + \sum, \; \ell, \; g, \; h \Big) = F_0' \; L', G', H' -, \; g + \sum_{j=1}^{\infty} \lambda^j \frac{\partial S_j}{\partial G'}$$

$$h' + \sum_{j=1} \lambda^j \frac{\partial S_j}{\partial H'} + \lambda F_1' \left(L', G', H' - g + \sum, \; h + \sum \right)$$

$$+ \lambda^2 F_2' \left(L', G', H' - g + \sum, \; h + \sum . \right) \tag{B}$$

Now expanding the left side of (B) in a Taylor series around L, G, H and F_1' and F_2' on the right hand side in a Taylor series around g', h' and keeping only terms up to and including the second order in λ, we have

$$F_0(L') + \frac{\partial F_0}{\partial L} \left[\frac{\partial S_1}{\partial \ell} \lambda + \frac{\partial S_2}{\partial \ell} \lambda^2 \right]$$

$$+ \lambda^2 F_1(L', G', H', \ell, g, h) + \lambda^2 \left[\frac{\partial F_1}{\partial L} \frac{\partial S_1}{\partial \ell} + \frac{\partial F_1}{\partial G} \frac{\partial S_1}{\partial g} + \frac{\partial F_1}{\partial H} \frac{\partial S_1}{\partial h} \right]$$

$$+ \lambda^2 F_2 \; [L', G', H', \ell, g, h] = F_0'(L', G', H' - g', h')$$

$$+ \lambda F_1'(L', G', H' - g, h) + \lambda^2 \left[\frac{\partial F_1'}{\partial g'} \frac{\partial S_1}{\partial G'} + \frac{\partial F_1}{\partial h'} \frac{\partial S_1}{\partial H'}\right]$$

$$+ \lambda^2 F_2' \; [L', G', H' - g, h] \qquad (C)$$

Now equating the coefficients of like powers of λ in (C) we obtain

$$F_0'(L', G', H' - g', h') = F_0(L') = \frac{U^2}{2L'^2} \qquad (D)$$

$$F_1(L', G', H', \ell, g, h) + \frac{\partial F_0}{\partial L} \frac{\partial S_1}{\partial \ell} = F_1'(L', G', H' - g, h) \qquad (E)$$

$$\frac{\partial F_0}{\partial L} \frac{\partial S_2}{F\ell} + \frac{\partial F_1}{\partial L} \frac{\partial S_1}{\partial \ell} + \frac{\partial F_1}{\partial G} \frac{\partial S_1}{\partial G'} + \frac{\partial F_1}{\partial H} \frac{\partial S_1}{\partial h}$$

$$+ F_2(L', G', H', \ell, g, h) = \frac{\partial F_1'}{\partial g'} \frac{\partial S_1}{\partial G'} + \frac{\partial F_1'}{\partial h'} \frac{\partial S_1}{\partial H'}$$

$$+ F_2'(L', G', H' - g, h). \qquad (F)$$

Now F_0' is determined by (D) so (E) may be solved by putting F_1' = part of F_1 independent of ℓ. This means $\left(\partial F_0/\partial L\right)\left(\partial S_1/\partial \ell\right) = -(F_1 - F_1') =$

- (part of F_1 dependent on ℓ). Now F_1' and S_1 are determined with the desired properties, and therefore (F) may be solved for S_2 by putting F_2' = part of F_2 independent of ℓ.

This determines F_2' and S_2 with the required properties. This process may be continued to determine F_n' and S_n to any desired order.

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